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MASS TRANSFER WITH A MOVING INTERFACE†

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Abstract—Mass transfer in the laminar, coccurent flow of two fluids in a rectangular channel is investigated. Distribution of concentrations and the local mass-transfer coefficients are calculated for the ideal flow, the Couette flow and for the flow with parabolic profile. Analytical and approximate solutions for finite and semi-infinite flows in a two-phase system are presented. The results obtained show that the formal mathematical solutions for one- and two-phase systems are similar. In the latter case, the diffusional resistance to the mass transfer is greater by a factor which is close to the value of the distribution coefficient.

NOMENCLATURE

A_n, constants defined by equations (24) and (41);
a, constant, equation (33);
a_k, constant defined by equations (6, 13) or (35);
b, constant, equation (33);
c, concentration;
D, diffusion coefficient;
D^*, transformed diffusion coefficient, equation (21);
d_k, transformed diffusion coefficient, equation (42);
d, constant, equation (37);
$F(\eta)$, function defined by equation (49);
h, length of the channel;
$i^n \operatorname{erfc} z$, repeated integrals of the error function;
$J_\nu(z)$, Bessel function;
k, local mass-transfer coefficient, defined by equation (14);
l_k, height of fluid in kth phase;
m, partition coefficient;
M_n, constant defined by equation (41);
Nu, diffusional Nusselt number, equation (48);
P, pressure;
p, constant defined by equations (13, 23) or order of Weber function;
p_n, eigenvalues, equation (43);
Q_k, function defined by equations (22) or (44);
q_k, constant defined by equation (5);
$R(z)$, function defined by equation (30);
$R(p, \alpha)$, function defined by equation (41);
$R_n(\rho)$, eigenfunctions, equation (20);
Re, Reynolds number;
r_k, constant equation (21);
$S(\rho)$, function defined by equation (25);
$T_\nu(\lambda|\rho)$, function defined by equation (25);
$T(\Omega|\eta)$, function defined by equation (35) or (37);
t_k, transformed variable, equation (21);
t_n, transformed variable, equation (42);
t, time;
U, velocity component in x direction;
$U^{\max}$, maximal velocity, equation (5);
$\bar{U}$, velocity, equation (6);
$\langle U \rangle$, average velocity, equation (8);
V, velocity component in y direction;
$W_e(p, t)$, even solution of Weber equation, equation (45);
$W_o(p, t)$, odd solution of Weber equation, equation (45);
w_k, transformed variable, equation (30);
X_n, eigenfunctions, equations (19) and (41);

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x, co-ordinate parallel to the direction of flow;
x^*, transformed variable, equation (42);
Y_n, eigenfunctions, equation (41);
y, co-ordinate perpendicular to the direction of flow;
y^*, transformed variable, equation (42);
Z, transformed variable, equation (13).

Greek symbols
α, γ, λ, eigenvalues, equations (22) and (42);
$\Gamma(z)$, gamma function;
δ, thickness of boundary layer, equations (18) and (48);
Δ, constant, equation (24);
η, $= Da^2x/U(0)^3$;
η, viscosity;
ν, kinematic viscosity, equation (48);
ξ, dimensionless co-ordinate, equation (5);
ξ, $= ay/U(0)$;
ρ_k, transformed variable, equation (21);
ρ, density, equation (47);
σ, constant, equation (30);
ϕ, $= (1 - \xi)(1 - 9\eta)^{\frac{1}{3}}$;
Ω, transformed variable, equation (35).

Subscripts and superscripts
k, phase;
i, component;
k, n, integers used for counting;
0, initial value.

INTRODUCTION

THE DESIGN of a separation process is usually based on theoretical predictions, which are derived from an idealized mathematical model. The present investigation considers the problem of mass transfer in the laminar, cocurrent flow of two liquids in a rectangular channel. The results of the ideal, the Couette and the parabolic flow models are presented. They are compared with the results of Beek and Bakker [1] for a one-phase system (Couette flow) and those of Tang and Himmelblau [2] for a two-phase system (flow with parabolic profile).

FORMULATION OF PROBLEM

In a two-phase (gas–liquid, liquid–liquid) separating element two immiscible fluids move between parallel planes. In the system under consideration, one component is distributed between the phases.

The mass transfer from one medium to the other is isothermal. Because hydrodynamic equilibrium is obtained more rapidly than diffusional, it is assumed that the steady-state distribution of velocities in the system is given by an independent solution of the hydrodynamic problem.

The velocity profile of the two-phase cocurrent laminar, stratified flow (two incompressiple, immiscible fluids of different viscosities and with equal heights) between parallel planes has been investigated by Bird, Stewart and Lightfoot [3]. Kapur and Shukla [4] generalized the solution for the case of n adjacent fluids of different heights. In the present work, the same results [4] will be applied, but in a somewhat different form which simplifies further calculations.

The physical situation considered is the parallel, steady-state flow of two stratified fluids of different heights, l_1 and l_2 (Fig. 1). In this case the Navier–Stokes equations are simplified to:

$$\eta_k \frac{\mathrm{d}^2 U_k}{\mathrm{d}y^2} = \frac{\mathrm{d}P}{\mathrm{d}x} \qquad k = 1, 2 \tag{1}$$

with the boundary conditions at the walls

$$U_1 = 0 \quad \text{at} \quad y = -l_1, \qquad U_2 = 0 \quad \text{at} \quad y = l_2 \tag{2}$$

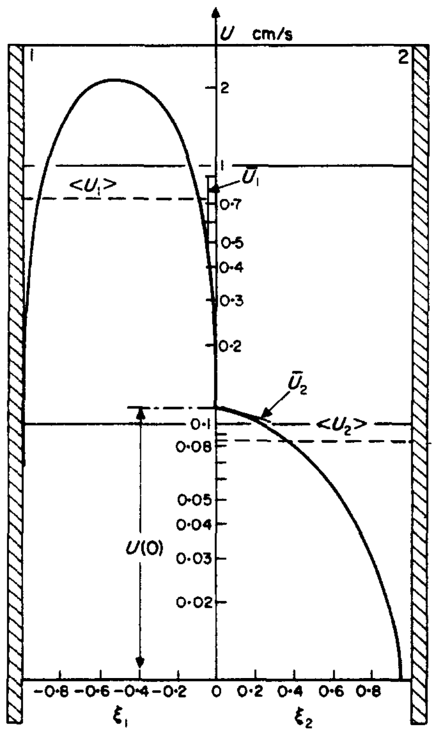

FIG. 1. Distribution of velocities in the separating element for a gas–liquid system.

and

$$\eta_1 \frac{dU_1}{dy} = \eta_2 \frac{dU_2}{dy} \quad \text{at} \quad y = 0, \qquad U_1 = U_2 \quad \text{at} \quad y = 0. \tag{3}$$

Equations (3) express the fact that the velocity and the momentum are continuous through the interface between two fluids. If it is assumed that the pressure gradient is constant, then the resulting velocity profile is parabolic:

$$U_k = U_k^{\max}\left[1 - \left(\frac{\xi_k + q_k}{q_k + (-1^k)}\right)^2\right] \qquad k = 1, 2 \tag{4}$$

where

$$\left.\begin{aligned} U_k^{\max} &= -\left(\frac{dP}{dx}\right)\frac{l_k^2}{\eta_k}[q_k + (-1)^k]^2 \\ q_k &= \frac{1}{2l_k}\left[\frac{l_1^2\eta_2 - l_2^2\eta_1}{l_1\eta_2 + l_2\eta_1}\right] \\ \xi_k &= y/l_k. \end{aligned}\right\} \tag{5}$$

In the neighbourhood of the interface (small values of y) where the interphase mass transfer occurs, the quadratic term of ξ in equation (4) can be neglected.

$$\left.\begin{aligned} \bar{U}_k(y) &= U(0) - a_k y, \qquad k = 1,2 \\ U(0) &= U_k^{\max}\left[1 - \left(\frac{q_k}{q_k + (-1)^k}\right)^2\right] \\ a_k &= U(0)\left(\frac{2q_k}{l_k[1 + (-1)^k 2q_k]}\right). \end{aligned}\right\} \tag{6}$$

The distribution of velocities obtained [equation (6)] may be considered to correspond to Couette flow, but with a moving interface. We will also use an approximation in which the actual velocity profile is replaced by the average velocity. The average velocity (indicated by dashed lines in Fig. 1) is defined as:

$$\langle U_k\rangle = \frac{1}{l_k^*}\int_0^{l_k^*} U_k(y)\,\mathrm{d}y, \qquad l_k^* = (-1)^k l_k. \tag{7}$$

From equation (4) and (7) one obtains

$$\langle U_k\rangle = -\left(\frac{\mathrm{d}P}{\mathrm{d}x}\right)\frac{l_k^2}{2\eta_k}\quad[\tfrac{2}{3} + (-1)^k q_k] \qquad k = 1,2. \tag{8}$$

The typical distribution of velocities in the separating element, for a gas–liquid and a liquid–liquid system, is plotted in Figs 1 and 2, respectively.

Applying the principle of mass conservation, the mass transfer (in the absence of chemical reaction) may be expressed as [3, 5, 6]:

$$\frac{\partial c_k}{\partial t} + U_k\frac{\partial c_k}{\partial x} + V_k\frac{\partial c_k}{\partial y} = D_k\left(\frac{\partial^2 c_k}{\partial y^2} + \frac{\partial^2 c_k}{\partial x^2}\right) \qquad k = 1,2 \tag{9}$$

where the diffusion coefficients, D_k, are assumed to be independent of concentrations. Equations (9) can be simplified, taking into consideration that the flow is parallel ($V_k = 0$). Additionally, the last term (molecular diffusion along the direction of the flow) is small and can be disregarded. In this way one obtains for a steady-state flow:

$$U_k(y)\frac{\partial c_k}{\partial x} = D_k\frac{\partial^2 c_k}{\partial y^2} \qquad k = 1,2. \tag{10}$$

The general solution of equations (10) with their boundary conditions is complicated. Three cases will be treated separately. The velocity profile, $U_k(y)$ will be replaced by $\langle U_k\rangle$, $\bar{U}_k$ and U_k [equations (6, 8, 4) respectively]. These cases will be called hereafter: the ideal flow approximation model, the Couette flow approximation model and the flow with parabolic velocity profile (parabolic model).

The following boundary conditions are adopted: the mass fluxes vanish at the walls, the interphase fluxes are equal and equilibrium exists at the interface. The separating element is continuously fed by a substance of uniform concentration.

$$\left.\begin{aligned} & c_k = c_k^0 \quad \text{at} \quad x = 0 \\ & -D_k\left(\frac{\partial c_k}{\partial y}\right) = 0 \quad \text{at} \quad y = l_k^* \\ & D_1\left(\frac{\partial c_1}{\partial y}\right) = D_2\left(\frac{\partial c_2}{\partial y}\right) \quad \text{at} \quad y = 0 \\ & c_1 = mc_2 \quad \text{at} \quad y = 0 \end{aligned}\right\} \tag{11}$$

where m is the distribution constant.

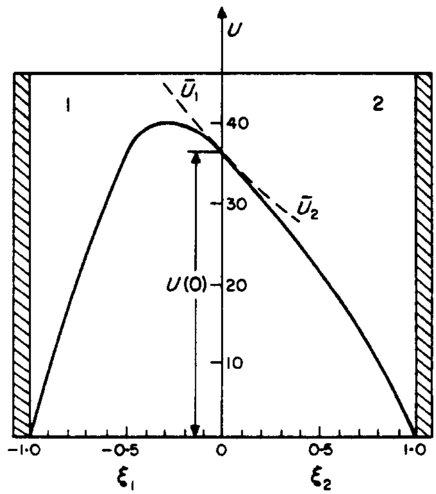

FIG. 2. Distribution of velocities in the separating element for a liquid–liquid system.

THE IDEAL FLOW APPROXIMATION MODEL

The distribution of concentrations in the case of the ideal flow model is given here only for the separating element, which consists of two-infinite media ($l_k \to \infty$)

$$\left.\begin{aligned} c_1(x, y) &= [(c_1^0 - mpc_2^0) + p(mc_2^0 - c_1^0)\operatorname{erf}|Z_1|]/(1-p) \\ c_2(x, y) &= [(c_1^0 - mpc_2^0) + (c_2^0 m - c_1^0)\operatorname{erf} Z_2]/(1-p)m \end{aligned}\right\} \tag{12}$$

where

$$\left.\begin{aligned} \operatorname{erf} Z &= \frac{2}{\sqrt{\pi}}\int_0^Z \exp(-t^2)\,\mathrm{d}t \\ Z_k &= y/2\left(\frac{x}{a_k}\right)^{\frac{1}{2}} \\ a_k &= \left(\frac{\langle U_k\rangle}{D_k}\right)^{\frac{1}{2}} \\ p &= \frac{D_2 a_2}{mD_1 a_1}. \end{aligned}\right\} \tag{13}$$

The mass transfer at the interphase (the local mass-transfer coefficient) is given by

$$k \equiv -D\left(\frac{\partial c}{\partial y}\right)\bigg|_{y=0} \tag{14}$$

and from (12)

$$k = \frac{c_1^0 - mc_2^0}{(\pi x)^{1/2}}\left[\frac{(\langle U_1\rangle D_1\langle U_2\rangle D_2)^{1/2}}{m(\langle U_1\rangle D_1)^{1/2} - (\langle U_2\rangle D_2)^{1/2}}\right]. \tag{15}$$

In this case, the mass transfer at the interphase differs by a numerical coefficient from the corresponding results of the penetration theory [3].

THE COUETTE FLOW APPROXIMATION MODEL

The steady-state distribution of concentrations in the separating element is obtained as a solution of the equations:

$$\bar{U}_k\frac{\partial c_k}{\partial x} = D_k\frac{\partial^2 c_k}{\partial y^2} \qquad k = 1, 2 \tag{16}$$

with the following boundary conditions

$$\left.\begin{aligned} &c_k(0, y) = c_k^0 \\ &\frac{\partial c_k}{\partial y} = 0 \qquad \text{at} \quad y = (-1)^k\,\delta_k \\ &D_1\left(\frac{\partial c_1}{\partial y}\right) = D_2\left(\frac{\partial c_2}{\partial y}\right) \qquad \text{at} \quad y = 0 \\ &c_1 = mc_2 \qquad \text{at} \quad y = 0 \end{aligned}\right\} \tag{17}$$

where δ_k is the thickness of the boundary layer in the kth phase. The value of δ can be estimated from Blasius solution for the flow along a very thin flat plate. It is convenient to express the boundary-layer thickness in the dimensionless form [7]:

$$\delta_k/h = 5/\sqrt{(Re_k)}. \tag{18}$$

In our case $U(0) \neq 0$ contrary to Blasius. It is therefore reasonable to introduce the "characteristic" velocity, $U_{\text{eff}} = U_k(\infty) + 1{\cdot}692U(0)$ into Reynolds number. This is done according to Acrivos' calculations [8]. Other workers proposed different relationships [1, 9]. The solution of equations (16) is:

$$c_k(x, \rho_k) = c_k^0 \sum_{n=1}^{\infty} A_n^{(k)}\, X_n^{(k)}(x)\, R_n^{(k)}(\rho_k) \tag{19}$$

where

$$\left.\begin{aligned} X_n^{(k)}(x) &= \exp\left[-\lambda_n^2 D_k^* x\right] \\ R_n^{(k)}(\rho_k) &= \rho_k^{1/2}\left[\frac{J_{2/3}(r_k\lambda_n)}{J_{-2/3}(r_k\lambda_n)} J_{1/3}(t_k) + J_{-1/3}(t_k)\right] \end{aligned}\right\} \tag{20}$$

and

$$\left.\begin{aligned} \rho_k &= a_k - y, \qquad t_k = \tfrac{2}{3}\lambda_n \rho_k^{\frac{3}{2}} \\ D_k^* &= D_k a_k/U(0), \\ r_k &= \tfrac{2}{3}[a_k - (-1)^k\, \delta_k]^{\frac{3}{2}}. \end{aligned}\right\} \tag{21}$$

According to equations (17) the eigenvalues of the problem, λ_n are given by the solution of equation

$$mD_1 a_1^{\frac{1}{3}} Q_1 = D_2 a_2^{\frac{1}{3}} Q_2 \tag{22}$$

where

$$\left.\begin{aligned} Q_k &= \frac{J_{-\frac{2}{3}}(p_k\lambda_n)\, J_{\frac{2}{3}}(r_k\lambda_n) - J_{-\frac{2}{3}}(r_k\lambda_n)\, J_{\frac{2}{3}}(p_k\lambda_n)}{J_{\frac{1}{3}}(p_k\lambda_n)\, J_{\frac{2}{3}}(r_k\lambda_n) + J_{-\frac{1}{3}}(p_k\lambda_n)\, J_{-\frac{2}{3}}(r_k\lambda_n)} \\ p_k &= \tfrac{2}{3}a_k^{\frac{3}{2}}, \qquad k = 1, 2. \end{aligned}\right\} \tag{23}$$

The coefficients of the Fourier–Bessel expansion are:

$$\left.\begin{aligned} A_n^{(k)} &= \frac{(-1)^k\, 6 T_{\frac{2}{3}}(\lambda_n | a_k)}{\lambda_n\{[S_n^{(1)}(\Delta_1) - S_n^{(1)}(a_1)] - [S_n^{(2)}(\Delta_2) - S_n^{(2)}(a_2)]\}} \\ \Delta_k &= [a_k - (-1)^k\, \delta_k] \end{aligned}\right\} \tag{24}$$

where

$$\left.\begin{aligned} T_\nu^{(k)}(\lambda_n | \rho_k) &\equiv \rho_k\left[J_\nu(t_k) - \frac{J_{\frac{2}{3}}(r_k\lambda_n)}{J_{-\frac{2}{3}}(r_k\lambda_n)}\, J_{\nu}(t_k)\right] \\ S_n^{(k)}(\rho_k) &\equiv [\rho_k R_{\frac{1}{3}}^{(k)}(t_k)]^2 - \rho_k T_{\frac{1}{3}}^{(k)}(\lambda_n | \rho_k)\, T_{\frac{2}{3}}^{(k)}(\lambda_n | \rho_k) \end{aligned}\right\} \tag{25}$$

The local mass-transfer coefficient at the interface is:

$$k = D_k c_k^0 \sum_{n=1}^{\infty} \frac{(-1)^k\, 6 a_k [T_{\frac{2}{3}}^{(k)}(\lambda_n | a_k)]^2\, X_n^{(k)}(x)}{[S_n^{(2)}(\Delta_2) - S_n^{(2)}(a_2)] - [S_n^{(1)}(\Delta_1) - S_n^{(1)}(a_1)]}. \tag{26}$$

It is evident from equations (19–26), that the obtained solution is complicated when the Couette flow approximation is assumed. Therefore, in order to obtain a solution, more convenient for engineering applications, we assume that the separating element consists of two semi-infinite media. Introducing a new variable, $\phi = (1 - \xi)(1/9\eta)^{\frac{1}{3}}$ where $\xi = ay/U(0)$, $\eta = Da^2x/U(0)$ the partial differential equations (16) are transformed into:

$$\frac{d^2 c_k}{d\phi_k^2} + 3\phi_k^2 \frac{dc_k}{d\phi_k} = 0, \qquad k = 1, 2 \tag{27}$$

with the corresponding boundary conditions

$$\left.\begin{aligned} & c_k = c_k^0 \quad \text{at} \quad \phi_k = \infty \\ & D_1^* \left(\frac{1}{9\eta_1}\right)^{\frac{1}{3}} \frac{dc_1}{d\phi_1} = D_2^* \left(\frac{1}{9\eta_2}\right)^{\frac{1}{3}} \frac{dc_2}{d\phi_2} \qquad \text{at} \quad \phi_k = \left(\frac{1}{9\eta_k}\right)^{\frac{1}{3}} \\ & c_1 = mc_2 \qquad \text{at} \quad \phi_k = (1/9\eta_k)^{\frac{1}{3}} \equiv w_k. \end{aligned}\right\} \tag{28}$$

The solution of equation (27) is:

$$\frac{c_k^0 - c_k}{c_1^0 - mc_2^0} = \sigma^{k-1} \frac{[1 - R(\phi_k)]}{[1 - R(w_1)] - m\sigma[1 - R(w_2)]} \qquad k = 1, 2 \tag{29}$$

where

$$\left.\begin{aligned} \sigma &= \left(\frac{D_1^2 a_1}{D_2^2 a_2}\right)^{\frac{1}{3}} \exp\left[\frac{U(0)^3}{9x}\left(\frac{a_1^2 D_1 - a_2^2 D_2}{a_1^2 a_2^2 D_1 D_2}\right)\right] \\ R(z) &= \frac{1}{\Gamma(\frac{4}{3})}\int_0^z \exp(-t^3)\,\mathrm{d}t. \end{aligned}\right\} \tag{30}$$

From equation (29), the local mass-transfer coefficient is:

$$k = \frac{c_1^0 - mc_2^0}{w_1\,\Gamma(\frac{4}{3})}\left[\frac{D_1^* \exp(-w_1^3)}{1 - R(w_1) - m\sigma(1 - R(w_2))}\right]. \tag{31}$$

Solution (29) is easy to use because values of the integral $R(z)$ were compiled by Abramowitz [10]. For small and large values of η the asymptotic expressions for equation (31) can be calculated, using the Beek–Bakker method [1]:

$$k = \frac{c_1^0 - mc_2^0}{(mD_1^*/D_2^*) - 1}\left[\frac{D_1^*}{4} + U(0)\left(\frac{D_1^*}{a_1 \pi x}\right)^{\frac{1}{2}}\right] \tag{32}$$

for $\dfrac{D_1^* a_1 x}{U(0)^2} \ll 1$

and

$$\left.\begin{aligned} k &= \frac{a(c_1^0 - mc_2^0)}{(mD_1^*/D_2^*) - 1}\left(\frac{D_1^* U(0)}{\sqrt{(a_1 x)}}\right)^{\frac{2}{3}}\left[1 + b\left(\frac{U(0)}{\sqrt{(D_1^* a_1 x)}}\right)^{\frac{2}{3}}\right] \\ a &= 3^{\frac{1}{3}}/\Gamma(\tfrac{1}{3}), \qquad b = 3^{\frac{1}{3}}(\Gamma(\tfrac{2}{3})/\Gamma(\tfrac{1}{3}))^2 \end{aligned}\right\} \tag{33}$$

for $\dfrac{D_1^* a_1 x}{U(0)^2} \gg 1.$

In a similar way, using the Laplace transform method [1], the asymptotic expressions for the distribution of concentrations in the separating element can be calculated for small and large values of η

$$\frac{c_k(\eta_k, \xi_k) - c_k^0}{c_1^0 - mc_2^0} = \left[\frac{(D_1^*/D_2^*)^{k-1}}{(mD_1^*/D_2^*) - 1}\right] T_1(\eta_k, \xi_k) \qquad k = 1, 2 \tag{34}$$

for $\eta_k \ll 1$

where

$$\left.\begin{aligned} T_1(\eta_k, \xi_k) &= (1 + \Omega_k)^{-\frac{1}{6}} \sum_{n=0}^{\infty} a_n^{(n)}(4\eta_k)^{n/2}\, i^n \operatorname{erfc} \frac{\Omega_k}{3\eta_k^{\frac{1}{2}}} \\ a_k^{(0)} &= 1, \qquad a_k^{(1)} = \frac{5}{48} \frac{\Omega_k}{1 + \Omega_k} \\ a_k^{(2)} &= -\frac{385\,(2 + \Omega_k)\,\Omega_k + 50\,(1 + \Omega_k)}{4608\,(1 + \Omega_k)^2} \\ \Omega_k &= \left|(1 - \xi_k)^{\frac{3}{4}} - 1\right| \end{aligned}\right\} \tag{35}$$

and for large values of η:

$$\frac{c_k(\eta_k, \xi_k) - c_k^0}{c_1^0 - mc_2^0} = \left[\frac{(D_1^*/D_2^*)^{k-1}}{(mD_1^*/D_2^*) - 1}\right] T_2(\eta_k, \xi_k) \qquad k = 1, 2 \tag{36}$$

where we denote

$$\left.\begin{aligned} T_2(\eta_k, \xi_k) &= \left\{(1 + \Omega_k)^{-\frac{1}{4}} + \frac{d}{\Gamma(\frac{2}{3})} \frac{1 - (1 + \Omega_k)^{\frac{3}{4}}}{(1 + \Omega_k)^{\frac{1}{4}}} \eta_k^{-\frac{1}{3}} - d^2 \frac{(1 + \Omega_k)^{\frac{1}{4}}}{\Gamma(\frac{1}{3})} \eta_k^{-\frac{2}{3}} + \dots\right\} \\ d &= 3^{\frac{1}{3}}\Gamma(\tfrac{2}{3})/\Gamma(\tfrac{1}{3}). \end{aligned}\right\} \tag{37}$$

The functions $T_1(\eta, \xi)$ and $T_2(\eta, \xi)$ [equations (35, 37)] can be treated as normalized distribution functions for the mass transfer, which occurs in two semi-infinite Couette flows with a moving interface. It is convenient to present these functions in the form $T(\Omega/\eta)$ taking Ω as a parameter. The graphs of $T(\Omega/\eta)$ for several values of Ω are given in Figs. 3 and 4. The behaviour of $T(\Omega/\eta)$ in the limit cases gives the expected results.

For $\Omega \to 0$, i.e. $y \to 0$ we have:

$$T_1(0|\eta) = 1 - \frac{25}{576}\eta + \dots \qquad \eta \ll 1 \tag{38}$$

when $\Omega \to \infty$, i.e. $y \to \infty$, $T_1(\infty|\eta) = 0$ and from (34) $c_k = c_k^0$.

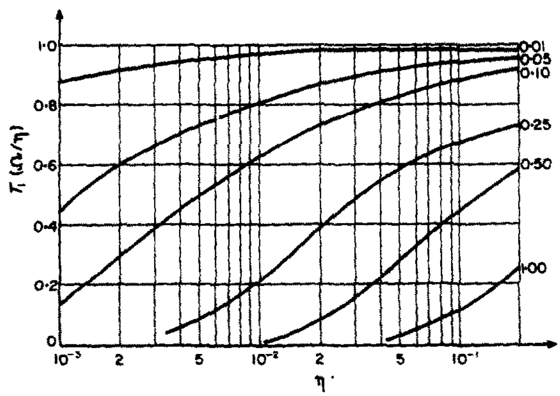

FIG. 3. Normalized distribution function for mass transfer [equation (35)].

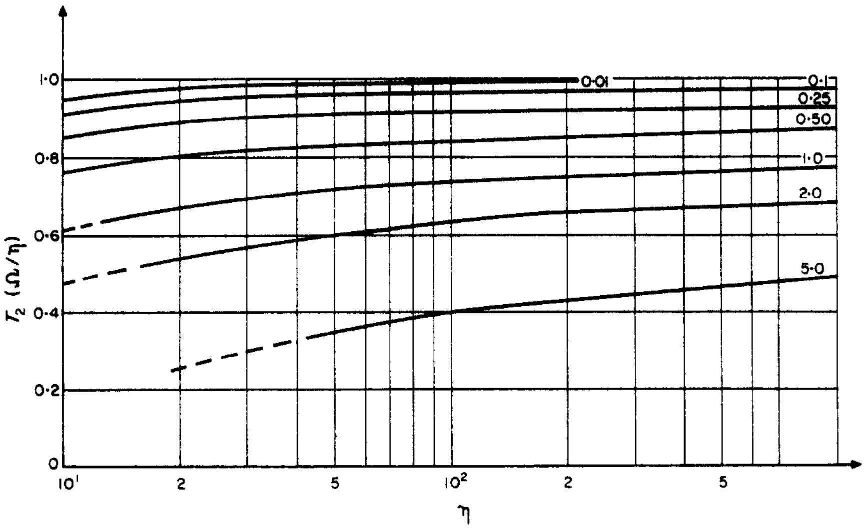

FIG. 4. Normalized distribution function for mass transfer [equation (37)].

Similary, when $\Omega = 0$

$$T_2(0\,|\,\eta) = 1 - \frac{d^2}{\Gamma(\frac{1}{3})\,\eta^{\frac{1}{3}}} + \ldots \qquad \eta \gg 1 \tag{39}$$

and $T_2(\Omega\,|\,\eta) \to 0$ for large Ω.

DISTRIBUTION OF CONCENTRATIONS IN THE FLOW WITH THE PARABOLIC VELOCITY PROFILE

Heat or mass transfer, which occurs in a fluid flowing through a channel with an established parabolic velocity profile, has been extensively studied [3, 5, 11, 12]. The results obtained for one-phase systems were applied to two-phase systems, assuming that the major resistance to mass transfer is present in one of the two phases. In the present work the distribution of concentrations in a two-phase system is given. The solution of equations (10) and (11) where U_k is given by (4) is

$$c_k(x_k^*, y_k^*) = c_k^0 \sum_{n=1}^{\infty} A_n^{(k)}\, X_n^{(k)}(x_k^*)\, Y_n^{(k)}(y_k^*) \qquad k = 1, 2 \tag{40}$$

where

$$\left.\begin{aligned}
A_n^{(k)} &= (-1)^k M_n^{-1} \int_{\alpha_n}^{\gamma_n^{(k)}} W(p_n, t_n^{(k)}\, \mathrm{d}t_n^{(k)}) \\
Y_n^{(k)}(y_k^*) &= R(p_n, \alpha_n)\, W_o(p_n, t_n^{(k)}) + W_e(p_n, t_n^{(k)}) \\
M_n &= md_1 \int_{\alpha_n}^{\gamma_n^{(1)}} [W(p_n, t_n^{(1)})]^2\, \mathrm{d}t_n^{(1)} + d_2 \int_{\gamma_n^{(2)}}^{\alpha_n} [W(p_n, t_n^{(2)})]^2\, \mathrm{d}t_n^{(2)} \\
X_n^{(k)}(x_k^*) &= \exp\left[-(2p_n + 1)^2\, x_k^*\right] \\
R(p_n, \alpha_n) &= \frac{2p_n W_o(p_n - 1, \alpha_n) - \alpha_n W_o(p_n, \alpha_n)}{\alpha_n W_e(p_n, \alpha_n) - 2p_n W_e(p_n - 1, \alpha_n)}
\end{aligned}\right\} \tag{41}$$

and

$$\left.\begin{aligned} x_k^* &= \frac{D_k x}{U_k^{\max}[l_k(q_k + (-1)^k)]^2}, & \alpha_n &= 2(p_n + \tfrac{1}{2})^{\frac{1}{2}} \\ y_k^* &= \frac{q_k + y/l_k}{[q_k + (-1)^k]}, & \gamma_n^{(k)} &= \alpha_n \frac{q_k}{[q_k + (-1)^k]} \\ d_k &= \frac{D_k}{l_k[q_k + (-1)^k]} & & \\ t_n^{(k)} &= \alpha_n y_k^*. & & \end{aligned}\right\} \tag{42}$$

The infinite set of eigenvalues $p_1, p_2, \ldots, p_n, \ldots$, is given by the solution of the characteristic equation:

$$m d_1 Q_1 = d_2 Q_2 \tag{43}$$

where

$$Q_k = \frac{R(p, \alpha)\, W_o'(p, \gamma^{(k)}) + W_e'(p, \gamma^{(k)})}{R(p, \alpha)\, W_o(p, \gamma^{(k)}) + W_e(p, \gamma^{(k)})} \tag{44}$$

and the even and odd solutions of Weber's equation are [13]:

$$\left.\begin{aligned} W_e(p, t) &= e^{-t^2/4}\left\{1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} p(p-2), \ldots, [p - 2(n-1)]\, t^{2n}\right\} \\ W_o(p, t) &= t\, e^{-t^2/4}\left\{1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (p-1)(p-3), \ldots, [p - (2n-1)]\, t^{2n}\right\} \\ W'(p, t) &= pW(p-1, t) - \frac{t}{2} W(p, t). \end{aligned}\right\} \tag{45}$$

Since in the characteristic equation (43) the argument and the order for Weber's functions contain p_n the obtained solution is much too complicated for practical use and asymptotic solutions are helpful.

The solution for small values of x^* cannot be obtained using the Laplace transform. This case will be treated in detail in a separate work, using the method proposed by Schlichting [7] and recently applied by Mercer [12] to the growth of the thermal boundary layer in the laminar flow between parallel flat plates. The solution of equations (10) and (11), when $l_k \to \infty$ and for large values of x^*, can be obtained using the Laplace transform method

$$\left.\begin{aligned} \frac{c_1(x, y) - c_1^0}{c_1^0 - mc_2^0} &= \frac{\left(\dfrac{q_1 - y/l_1}{q_1 - 1}\right)}{\left[1 - \dfrac{mD_1 q_1 l_2}{D_2 q_2 l_1}\left(\dfrac{q_2 + 1}{q_1 - 1}\right)^2\right]} \\ \frac{c_2(x, y) - c_2^0}{c_1^0 - mc_2^0} &= \frac{\left(\dfrac{q_2 + y/l_2}{q_2 + 1}\right)}{\left[m - \dfrac{D_2 q_2 l_1}{D_1 q_1 l_2}\left(\dfrac{q_1 - 1}{q_2 + 1}\right)^2\right]}. \end{aligned}\right\} \tag{46}$$

In this case ($x^* \gg 1$), the distribution of concentrations is independent of variable x. The concentration of the distributed substance varies linearly with y and increases or decreases up to the limit values.

DISCUSSION

The complex mechanism of mass transfer to stationary or mobile interfaces can only partly be described quantitatively. An accurate picture of mass transfer is difficult to give, because a separation process is influenced by many parameters. Detailed studies of hydrodynamic behaviour of gas–liquid and liquid–liquid interfaces has only recenly been undertaken and is still continuing. Experimentally it has been observed that the liquid surface is disturbed by wave motion and ripples. In liquid–liquid systems, a separation process is frequently accompanied by dispersing one immiscible liquid phase in the other in the form of drops. In this case, the mass transfer is complicated by deformation and displacement of the drops, which include linear and rotational motion, oscillations, surface waves, etc. [14]. In addition, since the accuracy of the experimental values of the local or the average mass-transfer coefficient is not great, there is no unique choice between existing theories of mass transfer. In the stagnant film theory [5], the mass-transfer coefficient k is proportional to the first power of the diffusion coefficient D. The boundary-layer theory [7] predicts an exponent $\frac{2}{3}$ and Higbie's penetration theory [5] predicts an exponent $\frac{1}{2}$. Theory and experiments show, that for mass transfer to mobile interfaces k varies with $D^{\frac{1}{2}}$ to $D^{\frac{2}{3}}$ [1, 3, 5, 11].

In engineering applications it is convenient to present the local mass-transfer coefficient k in the dimensionless form using the local diffusional Nusselt number. In our case it is possible to present the Nusselt number as the multiplication product of the function (which depends on coordinate only), and the factor, which characterizes the flow, properties of fluids and geometry of the system.

$$Nu_k = \frac{5\left(\dfrac{m\rho_2 Sc_2}{\rho_1 Sc_1}\right)^{k-1}}{\dfrac{m\rho_2 Sc_2}{\rho_1 Sc_1} - 1}\left(Sc_k \frac{U(0)}{U_{\text{eff}}^{(k)}}\right)^{\frac{1}{2}} F(\eta_k) \qquad k = 1, 2 \tag{47}$$

where

$$\left.\begin{aligned} Nu &\equiv \frac{k\,.\,\delta}{D(c_1^0 - mc_2^0)} \\ Sc &\equiv \frac{\nu}{D} \\ \delta &= 5\sqrt{\frac{\nu x}{U_{\text{eff}}}} \end{aligned}\right\} \tag{48}$$

and

$$F(\eta) = \begin{cases} \dfrac{1}{\sqrt{\pi}} + \dfrac{\eta^{\frac{1}{2}}}{4} & \text{for} \quad \eta \ll 1 \\ a\eta^{\frac{1}{6}}(1 + b\eta^{-\frac{1}{3}}) & \text{for} \quad \eta \gg 1. \end{cases} \tag{49}$$

The "reduced" mass-transfer function $F(\eta)$ is given for $10^{-2} < \eta < 10^3$ in Fig. 5. In limit cases: $F(\eta) \to \pi^{-\frac{1}{2}}$ (penetration theory) and $F(\eta)/a\eta^{\frac{1}{6}} \to 1$ when $\eta \to \infty$ (Lévêque solution [1]). In the neighbourhood of $\eta \sim 1$ the function can be interpolated between two given relationships (the dashed line in Fig. 5) or the local mass-transfer coefficient can be calculated exactly from a more complicated equation (31).

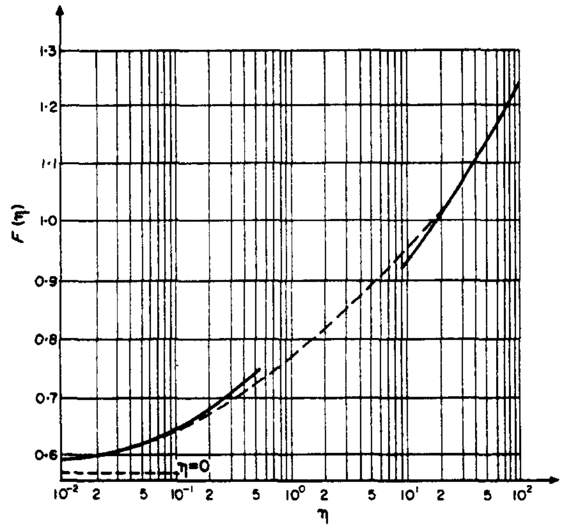

FIG. 5. The "reduced" mass-transfer function [equation (49)].

Relationships similar to the expression (49) were obtained by Beek and Bakker [1]. They treated the case of a semi-infinite Couette flow, bounded by a moving interface, across which mass transfer takes place. Comparing this case with the present one (two-semi-infinite Couette flows with a moving interface), it is evident that the formal solution in both cases is the same if $F(\eta)$ and the local Nusselt number for the one-phase problem are interchanged. In two-phase problems Nu contains also the parameters of the fluids (diffusion coefficient, viscosity, density), the geometry of the system and the distribution coefficient. However, as expected, the one-phase diffusional resistance in the one-medium system $1/k^*$, is lower than in the two-medium system $1/k$. Calculations for typical gas–liquid and liquid–liquid systems show that the ratio k/k^* is close to the value of the distribution coefficient m.

Similarly, the distribution of concentrations in two-phase systems can be presented in the form of the "normalized" distribution functions for mass transfer [equations (34) and (36)]. The function $T\ (\Omega(y)|\eta(x))$ is given for $10^{-3} < \eta < 10^{-1}$ in Fig. 3 and for $10 < \eta < 10^3$ in Fig. 4. $T_1(\Omega|\eta)$ varies strongly with η yet $T_2(\Omega|\eta)$ monotonically increases towards the limit values. The convergence of the series in equations (35) and (37) is rapid. For high values of η better accuracy is obtained from the analytical expression (29). The analytical solutions for the finite separating element [the Couette flow, equation (19) and the parabolic flow, equation (40)] are complicated, therefore the appropriate eigenvalue problems must be solved by a digital computer. In order to illustrate the results obtained in this work, the same system, studied by Tang and Himmelblau [2], was recalculated. Tang and Himmelblau investigated the interphase mass transfer for the laminar cocurrent flow of carbon dioxide and water between parallel plates. The absorption of gas into a liquid was analysed in a horizontal duct. A rectangular cross section of duct was chosen in order to simplify the mathematical analysis. The inside dimensions of the absorption channel were 2·22-cm high by 13·6-cm wide and of a total contact length 180 cm. The channel was half filled with water. In this experiment, the physical parameters were: $\langle U_2 \rangle = 0{\cdot}75$ cm/s, $c_2^0 = 0$, $D_2 = 1{\cdot}935 \,.\, 10^{-5}$ cm²/s, $D_1 = 0{\cdot}164$ cm²/s, $\eta_2 = 0{\cdot}894$ cP, $\eta_1 = 0{\cdot}0166$ cP and $m = 1{\cdot}64 \times 10^3$ (index 1 and 2 for CO_2 and water respectively). The eigenvalue problem in this case was solved numerically by Tang and Himmelblau on a digital computer using the Runge–Kutta method. The calculated concentration profile at the end of the test channel is presented in Fig. 6. The same example was

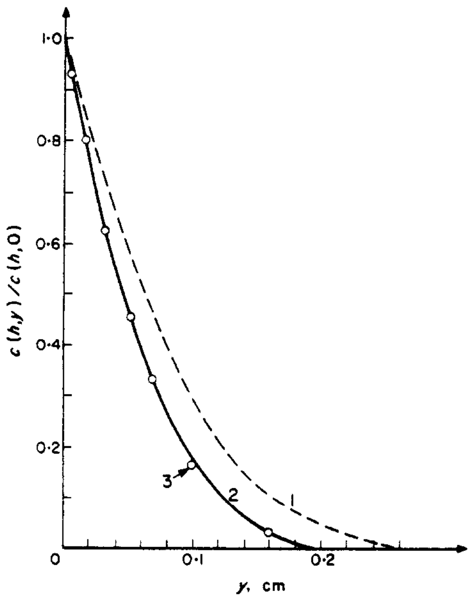

FIG. 6. Normalized concentration profile of CO_2 in water at the end of the test channel.
1—The ideal flow model
2—Calculations of Tang and Himmelblau [2]
3—The Couette flow model

recalculated according to the ideal flow model and the Couette flow model (in this case $\eta \ll 1$ and the function $T_1(\Omega/\eta)$ was used), and the results plotted in Fig. 6. The above calculations show that the Couette flow approximation model gives practically the same results as a parabolic flow model, yet the ideal flow model deviates in the positive direction of the mass transfer. However, in both cases the numerical solution is more simple to use than the solution of the eigenvalue problem. Finally, there is good agreement between the values of k calculated according to the boundary-layer theory, the penetration theory [2] and from equation (32) (the Couette flow model), ($k_{B-L} = 1{\cdot}965 \times 10^{-4}$, $k_p = 2{\cdot}095 \times 10^{-4}$ and $k_c = 2{\cdot}101 \times 10^{-4}$, at the end of the channel).

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Résumé—On étudie le transport de masse dans un écoulement laminaire de deux fluides à courants parallèles dans un canal rectangulaire. La distribution des concentrations et les coefficients locaux de transport de masse sont calculés pour l'écoulement idéal, l'écoulement de Couette et l'écoulement à profil parabolique. Des solutions analytiques et approchées pour des écoulements finis et semi-infinis dans un système à deux phases sont présentées. Les résultats obtenus montrent que les solutions mathématiques formelles sont semblables pour les systèmes à une phase et à deux phases. Dans le dernier cas, la résistance diffusionnelle au transport de masse est plus grande d'un facteur voisin de la valeur du coefficient de la distribution.

Zusammenfassung—Der Stoffübergang in einem rechteckigen Kanal, in dem zwei Flüssigkeiten mit einer gemeinsamen Grenzfläche strömen, wird untersucht. Die Konzentrationsverteilung und die lokalen Stoffübergangskoeffizienten werden für die ideale Strömung, die Couette-Strömung und für die Strömung mit parabolischem Profil berechnet. Analytische und angenäherte Lösungen für endliche und halbunendliche Strömungen in einem Zweiphasensystem werden aufgeführt. Die Ergebnisse, dass die formalen mathematischen Lösungen für Ein- und Zweiphasensysteme ähnlich sind. In letzterem Fall ist der Diffusionswiderstand für den Stoffübergang um einen Faktor grösser, der etwa dem Wert des Verteilungskoeffizienten entspricht.

Аннотация—Исследуется массообмен в ламинарном спутном потоке двух жидкостей в прямоугольном канале. Распределения концентрации и коэффициенты локального массообмена рассчитаны для идеального потока, потока Куэтта и потока с параболическим профилем. Приведены аналитические и приближенные решения для конечных и полубесконечных потоков в двухфазной системе. Полученные результаты показывают, что формальные математические решения для одно-и двухфазной систем являются автомодельными. В последнем случае диффузионное сопротивление массопереносу больше на порядок и приближается к значению коэффициента распределения.